STUDY MATERIAL (with assignment)
for
B.Sc. 2nd Semester, Statistics (Honours)
Paper: STA. HC. 2016, Unit: 2. (No. 1)

Joint Probability Distribution:

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(a) Discrete case:

For two discrete random variables X and Y taking values from the respective sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$, the joint probability $P(X=x_i, Y=y_j)$, that X takes the value x_i and Y takes the value y_j, simultaneously, is known as the joint probability distribution or the joint probability mass function of the two dimensional r.v. (X, Y).

Let $p_{ij} = P(x_i, y_j) = P(X=x_i, Y=y_j)$

then certainly we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} = 1$$

The joint probabilities p_{ij} can be written in table form as follows:

Joint Probability table.

X \ Y	y_1	y_2	$\cdots$	y_j	$\cdots$	y_n	Marginal total for X
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$	p_{1n}	$p_{1\bullet} = \sum_j p_{1j}$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	p_{2n}	$p_{2\bullet} = \sum_j p_{2j}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	p_{in}	$p_{i\bullet} = \sum_j p_{ij}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_m	p_{m1}	p_{m2}	$\cdots$	p_{mj}	$\cdots$	p_{mn}	$p_{m\bullet} = \sum_j p_{mj}$
Marginal total for Y	$p_{\bullet 1} = \sum_i p_{i1}$	$p_{\bullet 2}$	$\cdots$	$p_{\bullet j} = \sum_i p_{ij}$	$\cdots$	$p_{\bullet n}$	1

From the joint probability distribution, we can determine the probability distribution of variable X.

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$$P(X=x_i) = P(X=x_i \cap Y=y_1) + P(X=x_i \cap Y=y_2) + \cdots + P(X=x_i \cap Y=y_j) + \cdots + P(X=x_i \cap Y=y_m)$$

$$= \sum_{j=1}^{n} P(X=x_i \cap Y=y_j)$$

$$= \sum_{j=1}^{m} p_{ij}$$

$$= p_{i1} + p_{i2} + \cdots + p_{in}$$

$$= p_{i\bullet} \quad ; \quad i=1,2,\ldots,m.$$

This is known as the marginal probability distribution of X.

Clearly we have

$$\sum_{i=1}^{m} p_{i\bullet} = \sum_i \sum_j p_{ij} = 1$$

Similarly the marginal probability distribution of Y is

$$P(Y=y_j) = P(X=x_1 \cap Y=y_j) + P(X=x_2 \cap Y=y_j) + \cdots + P(X=x_m \cap Y=y_j)$$

$$= \sum_{i=1}^{m} p_{ij}$$

$$= p_{\bullet j} \quad , \quad j=1,2,\ldots,n$$

where $\sum_{j=1}^{n} p_{\bullet j} = 1$

Again the conditional probability

$$P(X=x_i / Y=y_j) = \frac{P(X=x_i \cap Y=y_j)}{P(Y=y_j)}$$

$$= \frac{P(x_i, y_j)}{P(y_j)} = \frac{p_{ij}}{p_{\bullet j}} \quad , \quad i=1,2,\ldots,m$$

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where $$\sum_{i=1}^{m} \frac{p_{ij}}{p_{\cdot j}} = \frac{\sum_{i=1}^{m} p_{ij}}{p_{\cdot j}} = \frac{p_{\cdot j}}{p_{\cdot j}} = 1$$

This is called the conditional probability distribution of X given $Y = y_j$.

Similarly the conditional distribution of Y given $X = x_i$ is defined as,

$$P[Y = y_j / X = x_i] = \frac{P(X = x_i \cap Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i\cdot}} \; ; \; j = 1, 2, \ldots, m.$$

where $$\sum_{j=1}^{n} \frac{p_{ij}}{p_{i\cdot}} = 1$$

<u>Independence of two random variables:</u>

Two r.v. X and Y are said to be independent if

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$$

or $$p_{ij} = p_{i\cdot} \times p_{\cdot j} \quad \forall (i, j)$$

In other words if the joint probability function is the product of the marginal probability functions then the two r.v. will be called independent

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Example:.

The joint probability distribution of (X, Y) is given below

X \ Y	-1	0	1
0	0	0·1	0·1
1	0·2	0·2	c
2	0	0·1	0·1

i) Find the value of c.
ii) Find marginal distributions of X and Y
(iii) Conditional distribution of X given $Y = 0$
(iv) Conditional distribution of Y given $X = 2$

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Solution: (i) We know that

Total probability $= 1$

$$\therefore \quad \sum_i \sum_j p_{ij} = 1$$

$$\Rightarrow 0 + 0.1 + 0.1 + 0.2 + 0.2 + c + 0 + 0.1 + 0.1 = 1$$

$$\Rightarrow \quad c = 1 - 0.8 = 0.2$$

We complete the joint probability table as follows:

X \ Y	−1	0	1	Marginal Totals for X $P_r(X=x)$
0	0	0.1	0.1	0.2
1	0.2	0.2	c = 0.2	0.6
2	0	0.1	0.1	0.2
Marginal Totals for Y $P_r(Y=y)$	0.2	0.4	0.4	1.0

(ii) The marginal distribution of X is

X	0	1	2	Total
$P_r(X=x)$	0.2	0.6	0.2	1

The marginal distribution of Y is

Y	−1	0	1	Total
$P_r(Y=y)$	0.2	0.4	0.4	1.0

(iii) The conditional probability distribution of X given $Y=0$

$$P(X=x_i / Y=y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)} = \frac{p_{ij}}{p_{\cdot j}}$$

Thus $$P(X=x_i / Y=0) = \frac{P(X=x_i, Y=0)}{P(Y=0)}, \quad x_i = 0, 1, 2$$

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When $X = 0$

$$P(X=0/Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)} = \frac{0.1}{0.4} = 0.25$$

when $X = 1$, $$P(X=1/Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{0.2}{0.4} = 0.5$$

when $X = 2$. $$P(X=2/Y=0) = \frac{P(X=2, Y=0)}{P(Y=0)} = \frac{0.1}{0.4} = 0.25$$

Thus the conditional distribution of X given $Y=0$ is

x	0	1	2	Total
$P(X=x/Y=0)$	0.25	0.5	0.25	1

Similarly the conditional distribution of Y given $X=2$ will be

y	-1	0	1	Total
$P(Y=y/X=2) = \frac{P(Y=y, X=2)}{P(X=2)}$	$\frac{0}{0.2} = 0$	$\frac{0.1}{0.2} = 0.5$	$\frac{0.1}{0.2} = 0.5$	1.0

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Assignment.

Answer the following questions.

1. Define joint probability, marginal distribution and conditional distribution of discrete random variables.
2. The following is the joint probability distribution of (X, Y).

$Y \backslash X$	1	2	3
-1	1/9	1/18	1/18
0	k	2/9	3/9
1	1/18	k	1/18

i) Find k.
ii) Find the marginal distributions of X & Y
(iii) Find the conditional dist^n of X given $Y=0$
(iv) Find the conditional dist^n of Y given $X=3$
(v) Evaluate $P(X < 3)$ and $P(Y \geq 0)$

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